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A Note on 2-fold Branched Covers of S^3

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1. Introduction

If a knot $K \subset S^3$ is amphicheiral, then the 2-fold covering space branched over K is symmetric (has an orientation reversing diffeomorphism). In this note we show that the converse is not true; it solves Problem 1.23 of [K], which is suggested by Montesinos [M1].

<u>Theorem.</u> There exist non-amphicheiral knots whose 2-fold branched covering spaces are symmetric.

In Section 2, we will construct a composite knot satisfying the properties of Theorem, and in Section 3, a prime knot.

2. A Composite Knot

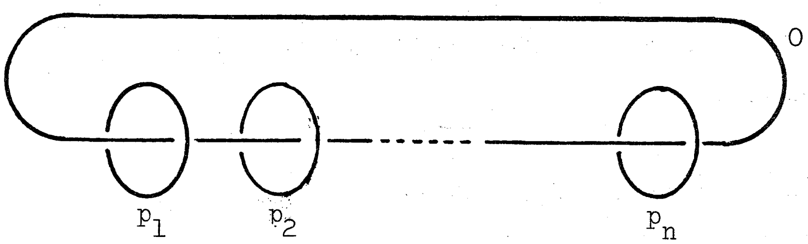
Let K_1 and K_2 be two knots. If there exists an orientation preserving homeomorphism of S^3 which maps K_1 onto K_2, then we will write $K_1 \approx K_2$.

<u>Lemma 1.</u> Let K_1 and K_2 be two prime knots such that $K_1 \not\approx r(K_2)$ where r is an orientation reversing homeomorphism of S^3. Then the

composite knot $K_1 \# K_2$ is amphicheiral if and only if both K_1 and K_2 are amphicheiral.

Proof. If $K_1 \# K_2$ is amphicheiral, then $K_1 \# K_2 \approx r(K_1 \# K_2) = r(K_1) \# r(K_2)$. It follows from the unique decomposition theorem [Sc] and $K_1 \not\approx r(K_2)$ that $K_1 \approx r(K_1)$ and $K_2 \approx r(K_2)$. The converse is clear.

Let K_1 be the pretzel knot (see [Tr] or [P]) of type $(p_1,\ldots,p_i,\ldots,p_j,\ldots,p_n)$, denoted by $K(p_1,\ldots,p_i,\ldots,p_j,\ldots,p_n)$, where n $(\geqq 5)$ is an odd integer and $p_1,p_2,\ldots,p_n$ $(\geqq 3)$ are distinct and odd integers. Let $K_2 = K(p_1,\ldots,p_j,\ldots,p_i,\ldots,p_n)$, where $i \neq j$. By the theorem in § 2 of [M1], the 2-fold covering spaces of S^3 branched over K_1 and K_2 are the same Seifert fiber space $(Ooo \mid 0;\ (p_1,1),\ (p_2,1),\ \ldots,\ (p_n,1))$ [Se], or the manifold with the following surgery presentation (see [R, Chapter 9]):



This manifold is prime by Theorem 7.1 and Lemma 10.2 of [W1]. Hence by the main result of [W2], K_1 and K_2 are prime knots. By the classification theorem of pretzel knots [P, p.56], we have $K_1 \not\approx K_2$. Since the signatures of K_1 and K_2 are both $n-1$ $(\neq 0)$ [P, p.71], K_1 and K_2 are non-amphicheiral [R, p.217]. It follows from Lemma 1 that $K_1 \# r(K_2)$ is non-amphicheiral.

On the other hand, the 2-fold covering space of S^3 branched over $K_1 \# r(K_2)$ is clearly symmetric. Therefore $K_1 \# r(K_2)$ is the desired composite knot.

Remark. We can also construct such a composite knot by using the examples of prime knots in [BGM] and [Ta] instead of pretzel knots.

3. A Prime Knot

Let M be the manifold which is obtained by Dehn surgery in the link L according to the diagram of Fig. 1. Since the figure-eight knot, which is the sublink of L, is amphicheiral, it is easily verified that M is symmetric. The link L is strongly-invertible because the symmetry with respect to the axis E leaves L invariant. Using the method of [M2], it can be shown that M is a 2-fold covering space branched over the knot K of Fig. 2. Because the signature of K is 16, K is non-amphicheiral. The following fact will be proved in Section 4.

Lemma 2. K is a prime knot.

4. Proof of Lemma 2

We will prove Lemma 2 by the method of Kirby and Lickorish [KL]. A tangle is a set of disjoint two arcs properly embedded in a 3-ball. The tangle as shown in Fig. 3 will be called a trivial tangle. A tangle T in a 3-ball B is prime if it has the following properties.

(i) Any 2-sphere in B, which meets T transversely in two points, bounds in B a ball meeting T in an unknotted spanning arc.

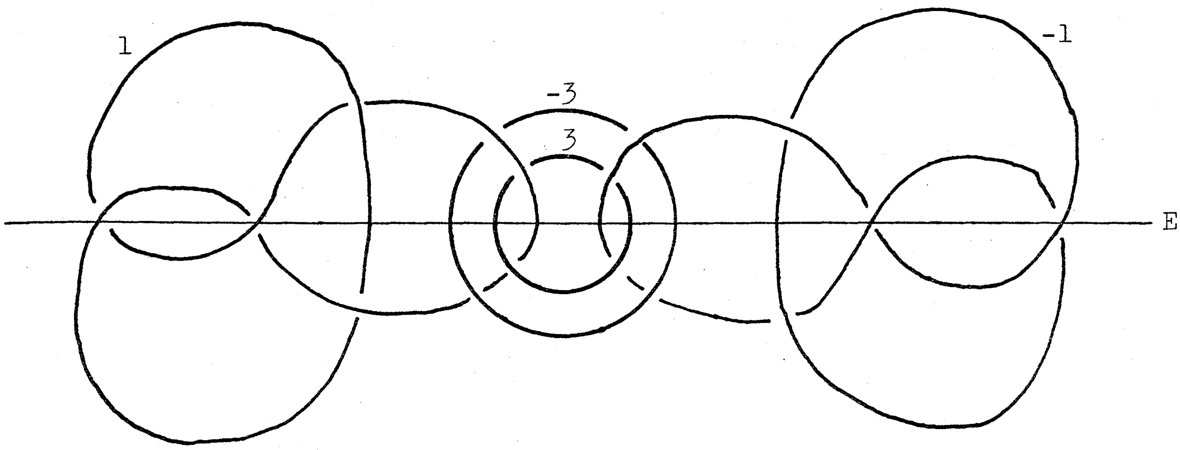


Fig. 1

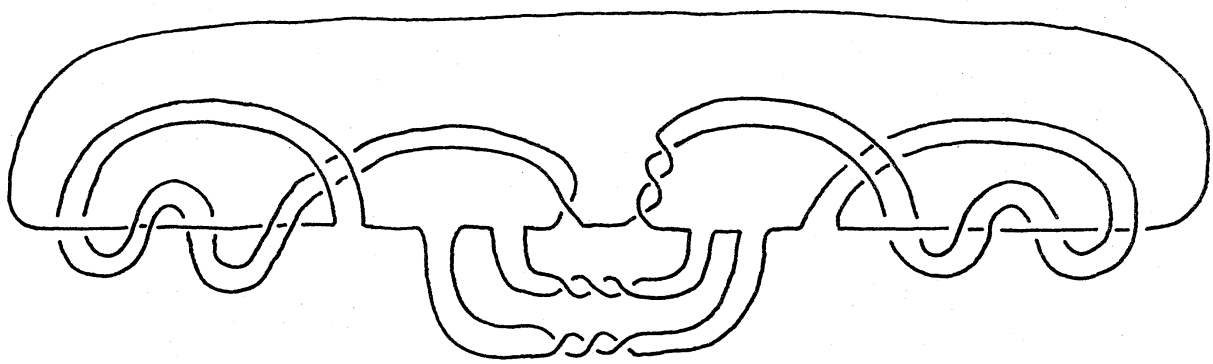


Fig. 2

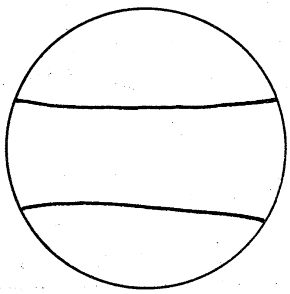


Fig. 3

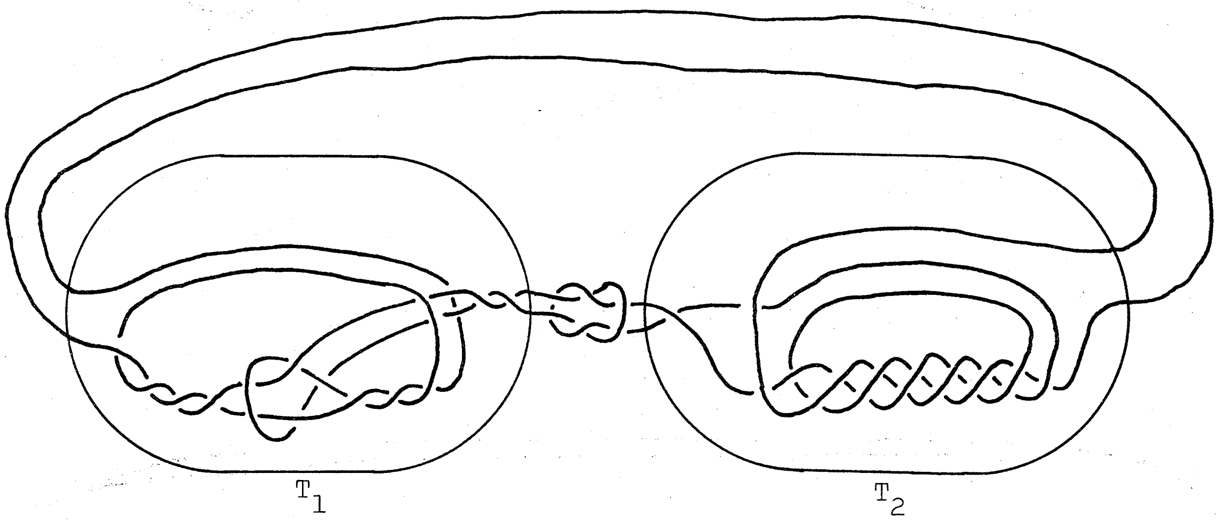


Fig. 4

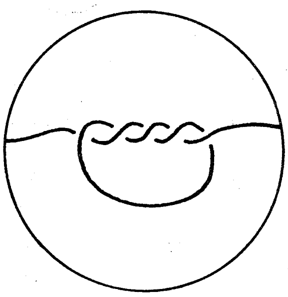


Fig. 5

(ii) The arcs of T cannot be separated by a disc properly embedded in B.

The knot K is constructed by the tangles T_1 and T_2 as shown in Fig. 4. To establish that K is prime, we have only to show that the tangles T_1 and T_2 are prime. If T_i $(i = 1,2)$ did not satisfy property (i) or property (ii), then there would be a ball meeting T_i in a knotted spanning arc as shown in Fig. 5. However, a trivial tangle can be added to T_i on the outside of the ball in Fig. 4 to create the knot K_i; K_1 is the pretzel knot $K(-2,3,7)$, and K_2 is the torus knot of type $(3,7)$, denoted by $T(3,7)$. Thus K_i would have $T(2,5)$ as a factor of its prime decomposition. Let $\Delta_i(t)$ and $\Delta(t)$ be the Alexander polynomials of K_i and $T(2,5)$ respectively, then $|\Delta_i(-1)| = 1$ must be divisible by $|\Delta(-1)| = 5$. This contradiction establishes Lemma 2.

<u>Remark.</u> K is concordant to $K(-2,3,7) \# T(3,7)$.

5. Question

Using the method of Viro [V], we can construct presumably non-amphicheiral knots; they are prime and the 2-fold covering spaces branched over them are symmetric. The knot of Fig. 6 is one of them. The 2-fold branched covering space is the manifold as shown in Fig. 7. Is this knot non-amphicheiral?

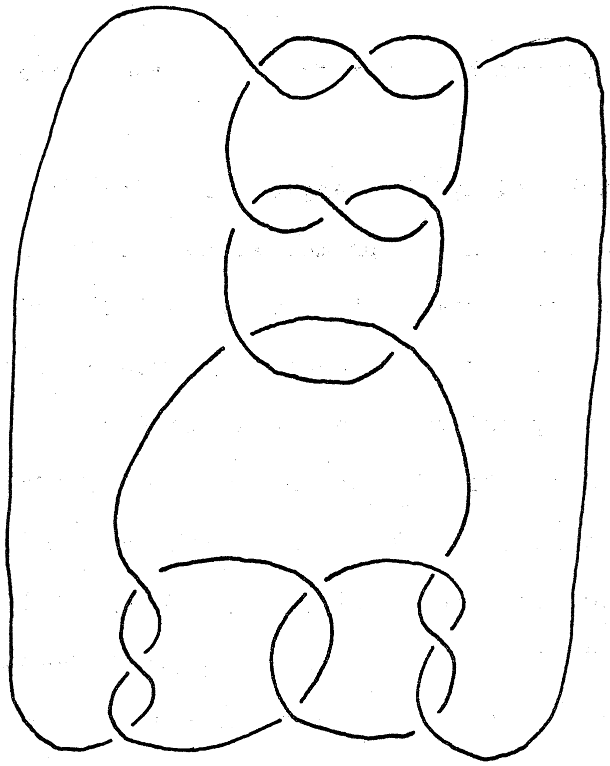


Fig. 6

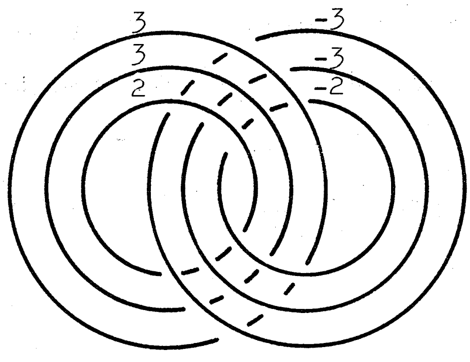


Fig. 7

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